Linear Algebra II

08/04/2015, Wednesday, 9:00 – 12:00

You are **NOT** allowed to use any type of calculators.

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1 (15 pts) **Gram-Schmidt process**

Consider the vector space $\mathbb{R}^4$ with the inner product

$$\langle x, y \rangle = x^T y.$$

Let $S \subset \mathbb{R}^4$ be the subspace given by

$$S = \text{span}\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \\ -1 \end{bmatrix} \right\}.$$

(a) Apply the Gram-Schmidt process to obtain an orthonormal basis for S.

(b) Find the closest element in the subspace S to the vector

$$\begin{bmatrix} a \\ b \\ b \\ a \end{bmatrix}$$

where a and b are real numbers.

2 (15 pts) **Cayley-Hamilton theorem**

Consider the matrix

$$M = \begin{bmatrix} 9 & -6 \\ 5 & -3 \end{bmatrix}.$$

By using the Cayley-Hamilton theorem, find a and b such that

$$M^5 = aM + bI.$$

3 $(2 + 8 + 5 = 15$ pts) **Singular value decomposition**

Consider the matrix

$$M = \begin{bmatrix} 1 & 2 & 3 \\ -1 & 2 & 3 \\ 1 & -2 & 3 \\ 1 & 2 & -3 \end{bmatrix}.$$

(a) Find the singular values of M.

(b) Find a singular value decomposition for M.

(c) Find the best rank 2 approximation of M.

4 $(5+5+5=15$ pts) **Eigenvalues and singular values**

(a) Let A be a square matrix. Show that

$$\det(e^A) = e^{\mathrm{tr}(A)}.$$

(b) Let B be an orthogonal matrix. Find out the singular values of B.

(c) Let C and D be $n \times n$ matrices. Suppose that C is orthogonal. Find out the relationship between the singular values of D and those of CD.

5 $(10+5=15$ pts) **Positive definiteness**

(a) Consider the function

$$f(x,y) = 6xy^2 - 2x^3 - 3y^4.$$

Find the stationary points of f and determine whether its stationary points are local minimum/maximum or saddle points.

(b) Let

$$M = \begin{bmatrix} 2 & 1 & a \\ 1 & 2 & 1 \\ a & 1 & 2 \end{bmatrix}$$

where a is a real number. Determine all values of a for which M is

(i) positive definite.

(ii) negative definite.

6 $(2+3+10=15$ pts) **Jordan canonical form**

Consider the matrix

$$M = \begin{bmatrix} -1 & 1 & 0 \\ -1 & 0 & 1 \\ -1 & 0 & 1 \end{bmatrix}.$$

(a) Find the eigenvalues of M.

(b) Is M diagonalizable? Why?

(c) Put M into the Jordan canonical form.

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10 pts free